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Some graph theoretic aspects of finite geometries

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There are a couple of graph theoretical questions for which finite geometries provide answers or settings in which they become particularly interesting. On the other hand, the research motivated by these problems support the development of finite geometry. In the current thesis we wish to show some results given birth to by such interactions between the two fields; however, the emphasis is on the finite geometrical viewpoint. Two of the examined areas, the problem of cages and the Zarankiewicz problem, have extremal combinatorial origins, and have attracted a considerable amount of interest since the middle of the last century. It is well-known that generalized polygons play a prominent role in some particular cases of the cage-problem, just as designs do in the Zarankiewicz problem. The idea we consider in this work regarding these problems is that the known extremal structures contain substructures that provide us other valuable constructions. We mainly investigate projective planes, which can be considered as generalized triangles and as symmetric 2-designs as well, but the more general cases are also touched. Besides the above problems, we also consider two general notions from graph theory in the setting of finite projective planes. The concept of semi-resolving sets is motivated by localizational questions in graphs, and their study for projective planes was proposed by R. F. Bailey. In the theory of hypergraph coloring, the upper chromatic number is the counterpart of the classical chromatic number in some sense, and its study has been intensively encouraged by V. Voloshin. Projective planes, considered as hypergraphs, are naturally arising examples to investigate this problem for, and their study was started in the mid-nineties.

As for the techniques, the polynomial method plays a crucial role in some of the proofs. We use Rédei polynomials, the newly developed Szőnyi–Weiner Lemma, and the multiplicity version of Alon's combinatorial Nullstellensatz. For the sake of completeness, we give the proof of the middle one in the Appendix.

Now let us introduce the main results of the thesis. The missing basic facts about finite projective planes can be found in [10]. We denote by Π_q an arbitrary projective plane of order q, while $\mathrm{PG}(2,q)$ and $\mathrm{AG}(2,q)$ stand for the projective and affine planes over $\mathrm{GF}(q)$ (the finite field of q elements), respectively. The point-set and the line-set of the point–line geometry in question are denoted by $\mathcal{P}$ and $\mathcal{L}$, respectively. A line ℓ is called a t-secant to a pointset $\mathcal{B}$ if $|\ell \cap \mathcal{B}| = t$.

Chapter 2 is devoted to results on multiple blocking sets, which are closely related to three out of the four main problems of the thesis. Blocking sets and multiple blocking sets are intensively studied objects in finite geometry.

Definition 1.5.3. *A point-set* $\mathcal{B}$ *in* $\Pi_q = (\mathcal{P}, \mathcal{L})$ *is a* t-fold blocking set *if* $|\ell \cap \mathcal{B}| \geq t$ *for all* $\ell \in \mathcal{L}$. *A line-set* $\mathcal{S}$ *in* Π_q *is a* t-fold covering set *if there are at least* t *lines of* $\mathcal{S}$ *through all* $P \in \mathcal{P}$. *Note that this is the dual of a t-fold blocking set. One-fold and two-fold blocking (covering) sets are also called a* blocking set *(*covering set*) and a* double blocking set *(*double covering set*), respectively.*

Definition 1.5.4. *Let* $\mathcal{B}$ *be a t-fold blocking set in* $\Pi_q = (\mathcal{P}, \mathcal{L})$. *A point* $P \in \mathcal{B}$ *is* essential, *if* $\mathcal{B} \setminus \{P\}$ *is not a t-fold blocking set; equivalently, if there is a t-secant to* $\mathcal{B}$ *through* P. $\mathcal{B}$ *is* minimal, *if all its points are essential; equivalently, if it does not contain a smaller t-fold blocking set.*

Definition 1.5.5. *The size of the smallest t-fold blocking set in* Π_q *is denoted by* $\tau_t(\Pi_q)$, *and it is called the* t-blocking number *of* Π_q.

For example, a Baer subplane (a subplane of order $\sqrt{q}$) is a blocking set. Note that by duality, if we have a result on (multiple) blocking sets, then we also have the dual result for (multiple) covering sets. We prove the following theorem.

Theorem 2.1.1. *Let* $\mathcal{B}$ *be a t-fold blocking set in* $\mathrm{PG}(2,q)$, $|\mathcal{B}| = t(q+1)+k$. *Then there are at least* $q+1-k-t$ *distinct t-secants to* $\mathcal{B}$ *through any essential point of* $\mathcal{B}$.

Corollary 2.1.3. *Let* $\mathcal{B}$ *be a t-fold blocking set in* $\mathrm{PG}(2,q)$ *with* $|\mathcal{B}| \leq (t+1)q$ *points. Then there is exactly one minimal t-fold blocking set in* $\mathcal{B}$, *namely the set of essential points of* $\mathcal{B}$.

Corollary 2.1.3 is in fact equivalent with Theorem 2.1.1, and it also follows from a recent, more general result of Harrach [9]. We also give a description of a somewhat artificially looking type of almost covering sets, which we need later.

Lemma 2.1.6. *Let* $\mathcal{L}$ *be a set of non-vertical lines of* $\mathrm{AG}(2,q)$, q *a power of the prime* p, *which cover every point of* $\mathrm{AG}(2,q)$ *exactly* k *times* $(k \geq 1)$ *except possibly the points of* ν *fixed vertical lines, where* $\nu(k+1) \leq q$ *and* $\nu k < p$. *Then* $\mathcal{L}$ *consists of the union of* k *parallel classes, or* $\mathcal{L}$ *consists of the* kq *non-vertical lines passing through* k *fixed points on a fixed vertical line. (Note that the two possibilities are basically the same.)*

If q is a square, then $\mathrm{PG}(2,q)$ is well-known to be partitionable into $(q-\sqrt{q}+1)$ pairwise disjoint Baer subplanes, hence $\tau_t(\mathrm{PG}(2,q)) \leq t(q+\sqrt{q}+1)$ for all $1 \leq t \leq q-\sqrt{q}$. This is known to be the smallest possible construction if t is small.

If $t \geq 2$ and q is not a square, no general construction for small t-fold blocking sets were known. We construct a small double blocking set in $\mathrm{PG}(2,q)$ for each $q = p^h$ as the union of two disjoint blocking sets, where p and $h \geq 3$ are odd.

THEOREM 2.2.1. *Let $h \geq 3$ odd, $\alpha \geq 1$ an integer, p an odd prime, $r = p^\alpha$, $q = r^h$. Then there exist two disjoint blocking sets of size $q + (q-1)/(r-1)$ in* $\mathrm{PG}(2,q)$. *Consequently,* $\tau_2(\mathrm{PG}(2,q)) \leq 2(q + (q-1)/(r-1))$.

Let us mention that a result by Blokhuis, Storme, and Szőnyi [5] rouglhy says that a double blocking set in $\mathrm{PG}(2,q)$ of size at most $2q + q^{2/3}$ contains the union of two disjoint Baer subplanes. Theorem 2.2.1 shows that the term $q^{2/3}$ is of the right magnitude if q is a cube. In the proof of Theorem 2.2.1 we need the following variation of the Hermite–Dickson Theorem on permutation polynomials.

THEOREM 2.2.4. *Let* $\mathrm{GF}(q)$ *be a field of characteristic p, let $m \mid q-1$, and let D be the multiplicative subgroup of* $\mathrm{GF}(q)^*$ *of m elements. Let $g \in$* $\mathrm{GF}(q)[x]$ *be a polynomial such that $g(b) \in D$ for all $b \in D$. Then $g|_D \colon D \to D$ is a permutation of D if and only if the constant term of $g(x)^t \pmod{x^m - 1}$ is zero for all $1 \leq t \leq m-1$, $p \nmid t$.*

In $\mathrm{PG}(2,q)$ there are tight lower bounds on the size of a multiple blocking set. For arbitrary projective planes, Ball [4] has the following result.

RESULT 2.3.1 (BALL [4]). *Let $\mathcal{B}$ be a t-fold blocking set in an arbitrary projective plane of order q that contains no line, and assume $1 \leq t \leq q-2$. Then*

$$|\mathcal{B}| \geq tq + \sqrt{tq} + 1.$$

We give the proof of Ball in a more compact formulation using the standard equations, and also present a slight improvement for $t \geq 2$.

REMARK 2.3.2. *Let $\mathcal{B}$ be a t-fold blocking set in an arbitrary projective plane of order q that contains no line, and assume $1 \leq t \leq q-3$. Then*

$$|\mathcal{B}| \geq tq + \sqrt{tq} + \frac{t}{2}\left(1 - \sqrt{\frac{t}{q}}\right) + \frac{1}{2}.$$

We also derive a lower bound for multiple blocking sets that may contain lines.

THEOREM 2.3.3. *Let $\mathcal{B}$ be a t-fold blocking set in an arbitrary projective plane of order q, $2 \leq t \leq q-3$. Then*

$$|\mathcal{B}| \geq tq + \sqrt{(t-1)q} - t + 3.$$

In Chapter 3 we study a particular case of a classical problem of extremal graph theory. The girth of a graph is the length of the shortest cycle in it. A (k, g)-graph is a k-regular graph of girth g. Such graphs exist if $k \geq 2$ and $g \geq 3$ [7]. A (k, g)-graph is called a cage graph, if it has the minimum possible number of vertices among (k, g)-graphs. Determining the order $c(k, g)$ of (k, g)-cages for arbitrary parameters seems hopeless. Note that the cases $k = 2$ and $g \leq 4$ are trivial. A well-known, easy lower bound for $c(k, g)$ is the following.

Result 3.1.2 (Moore bound).

$$c(k,g) \geq c_0(k,g) = \begin{cases} 1 + \sum\limits_{i=0}^{(g-3)/2} k(k-1)^i & \textit{if } g \textit{ is odd,} \\ 2 \sum\limits_{i=0}^{(g-2)/2} (k-1)^i & \textit{if } g \textit{ is even.} \end{cases}$$

A (k, g)-graph on $c_0(k, g)$ vertices is called a Moore graph or a Moore cage. If g is fixed, sequences of cages for infinitely many ks are known only in particular cases.

Definition 1.4.1 (Generalized polygon, GP). *Let $n \geq 3$. A point-line incidence structure is a* generalized n-gon of order (s, t) *if and only if the following hold:*

GP1: every point is incident with $s + 1$ lines;

GP2: every line is incident with $t + 1$ points;

GP3: the diameter and the girth of its incidence graph are n and $2n$, respectively.

If $s = t = q$, then we say that the GP is of order q.

It is well-known that $(q + 1, 2n)$ Moore cages are the incidence graphs of generalized n-gons of order q, which exist only if $n = 3, 4, 6$; in these cases there are constructions if q is a power of a prime. Generalized triangles of order $q \geq 2$ and projective planes of order q are the same. In Chapter 3 we consider the idea of constructing small $(k, 2n)$-graphs as subgraphs of the known $(q + 1, 2n)$-cages, $k \leq q$. In Section 3.5 we show that different constructions of various authors (e.g., [1, 6] and particular cases of [11]) can be unified with this concept. To obtain an induced regular subgraph of a regular graph, we need to delete the following substructure.

DEFINITION 1.2.3. *Let $G=(V;E)$ be a graph. A vertex-set $S \subsetneq V$ is called a* t-fold perfect dominating set *(abbreviated as t-PDS), if all $v \in V \setminus S$ has precisely t neighbors in S. A 1-PDS is also called a perfect dominating set (PDS).*

It is clear that t-PDSs and $(q+1-t)$-regular induced subgraphs of the incidence graph of a generalized polygon are in a one-to-one correspondence. To obtain the smallest $(q+1-t)$-regular induced subgraph for a fixed t, we need to find the largest t-PDS. In a projective plane a t-PDS is a pair $\mathcal{T}$ of a point-set $\mathcal{P}_0$ and a line-set $\mathcal{L}_0$ such that every point not in $\mathcal{P}_0$ is covered by exactly t lines of $\mathcal{L}_0$, and dually, every line not in $\mathcal{L}_0$ contains exactly t points of $\mathcal{P}_0$. It is easy to see that for any t-PDS $\mathcal{T}=(\mathcal{P}_0,\mathcal{L}_0)$, $|\mathcal{P}_0|=|\mathcal{L}_0|$ holds; thus the size of $\mathcal{T}$ is defined to be $|\mathcal{P}_0|$. There is one general t-PDS known that works for arbitrary t and all generalized n-gons of order q, which has size $tq^{n-2}+\sum_{i=0}^{n-3} q^{n-3-i}$. This construction was found independently by several authors. This yields $c(k,2n) \leq 2kq^{n-2}$, where q is the smallest prime power larger than or equal to k and $n \in \{3,4,6\}$. Note that, combined with the Moore bound, this implies $c(k,2n) \sim 2k^{n-1}$ in these cases. Although we have some specific constructions for generalized quadrangles as well, we focus on the case of projective planes below. The most important two constructions of t-PDSs in projective planes are the following. For a point P or a line ℓ, let $[P]$ and $[\ell]$ denote the set of lines incident with P and the set of points incident with ℓ, respectively.

CONSTRUCTION 3.3.1. Let $\mathcal{P}^*=\{P_1,\ldots,P_t\}$ and $\mathcal{L}^*=\{\ell_1,\ldots,\ell_t\}$ be a proper point-set and a proper line-set of Π_q, respectively, such that every line connecting two points of $\mathcal{P}^*$ is in $\mathcal{L}^*$, and the intersection point of any two lines of $\mathcal{L}^*$ is in $\mathcal{P}^*$ (that is, $(\mathcal{P}^*,\mathcal{L}^*)$ is a (possibly degenerate) subplane). Let $\mathcal{P}_0=\mathcal{P}^* \bigcup \cup_{i=1}^{t}[\ell_i]$, $\mathcal{L}_0=\mathcal{L}^* \bigcup \cup_{i=1}^{t}[P_i]$. Then $\mathcal{T}=(\mathcal{P}_0,\mathcal{L}_0)$ is a t-PDS. The largest t-PDS obtained in this way is of size $q+2$ for $t=1$ and $tq+1$ for $t \geq 2$.

CONSTRUCTION 3.3.2. Let $\mathcal{B}_i=(\mathcal{P}_i,\mathcal{L}_i)$, $i=1,\ldots,t$, be t mutually disjoint Baer subplanes, $\mathcal{P}_0=\cup_{i=1}^{t}\mathcal{P}_i$, $\mathcal{L}_0=\cup_{i=1}^{t}\mathcal{L}_i$. Then $\mathcal{T}=(\mathcal{P}_0,\mathcal{L}_0)$ is a t-PDS of size $t(q+\sqrt{q}+1)$.

We prove the following results. Note that as usually there is a prime between q and $q-\sqrt{q}$, we find a $(k,6)$ Moore cage with $q-\sqrt{q} \leq k \leq q$, so this construction method is useful mostly for $t \leq \sqrt{q}$.

THEOREM 3.3.5. *Let $\mathcal{T}=(\mathcal{P}_0,\mathcal{L}_0)$ be a t-PDS in Π_q, and suppose $t \leq 2\sqrt{q}$. Then $|\mathcal{T}| \leq t(q+\sqrt{q}+1)$.*

Note that by the above theorem, Construction 3.3.2 is optimal. The upcoming theorems show that if t is not too large, then there are no other constructions than Constructions 3.3.1 and 3.3.2.

THEOREM 3.3.7. *Every PDS of any finite projective plane is either Construction 3.3.1 or 3.3.2 (with $t = 1$).*

THEOREM 3.3.10. *Let $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ be a 2-PDS in* $\mathrm{PG}(2, q)$.

1. *If $q \geq 9$, then $\mathcal{T}$ is either Construction 3.3.1 (with $t = 2$), or $|\mathcal{T}| = 2(q + \sqrt{q} + 1)$.*
2. *If $q > 256$, then $\mathcal{T}$ is either Construction 3.3.1 or 3.3.2 (with $t = 2$).*

THEOREM 3.3.11. *Let p be a prime and let $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ be a t-PDS in* $\mathrm{PG}(2, q)$, *$q = p^h$; furthermore,*

- *for $h = 1$ and $h = 2$, let $t < p^{1/2}/2$;*
- *for $h \geq 3$, let $t < \min\{p + 1, c_p q^{1/6} - 1, q^{1/4}/2\}$, where $c_2 = c_3 = 2^{-1/3}$ and $c_p = 1$ for $p > 3$.*

Then $\mathcal{T}$ is either Construction 3.3.1 or 3.3.2.

Finally, we end Chapter 3 by showing a small non-induced regular subgraph of generalized n-gons, $n = 3, 4, 6$. Recall that the only known general construction for induced subgraphs yields $c(k, 2n) \leq 2kq^{n-2}$.

THEOREM 3.6.1. *Suppose that a generalized n-gon of order q exists, and let $3 \leq k \leq q$. If $n \geq 4$, then $c(k, 2n) \leq 2k(q^{n-2} - q^{n-4})$; if $n = 3$, then $c(k, 2n) \leq 2kq - 2$.*

A resolving set for a graph G is a subset S of its vertices such that every vertex of G can be identified by its distance list with respect to S; that is, $(d(v, s_1), \ldots, d(v, s_k))$ is unique for every vertex v of G, where $d(x, y)$ denotes the distance of two vertices x and y, and $S = \{s_1, \ldots, s_k\}$. A semi-resolving set for a bipartite graph G with vertex classes A and B is a subset of, say, A such that every vertex of B can be identified by its distance list with respect to S. R. F. Bailey [3] proposed the study of resolving and semi-resolving sets for the incidence graphs of projective planes. In Chapter 4 we study the latter and show the following. Let $\mu_S(\Pi)$ be the size of the smallest semi-resolving set for the incidence graph of the projective plane Π.

PROPOSITION 4.1.4. *Let Π_q be an arbitrary projective plane. Then*

(i) $\mu_S(\Pi_q) \leq \tau_2(\Pi_q) - 1$ *(also pointed out by Bailey);*

(ii) if $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are disjoint blocking sets in* Π_q*, then* $\mu_S(\Pi_q) \leq |\mathcal{B}_1| + |\mathcal{B}_2| - 2$*;*

(iii) in particular, if q *is a square prime power, then* $\mu_S(\mathrm{PG}(2,q)) \leq 2q + 2\sqrt{q}$*; if* $q = r^h$*,* $h \geq 3$ *odd,* r *odd, then* $\mu_S(\mathrm{PG}(2,q)) \leq 2(r^h + r^{h-1} + \ldots + r)$*.*

THEOREM 4.1.10. *Let* $\mathcal{S}$ *be a semi-resolving set for* $\mathrm{PG}(2,q)$*,* $q \geq 4$*. If* $|\mathcal{S}| < 9q/4 - 3$*, then one can add at most two points to* $\mathcal{S}$ *to obtain a double blocking set.*

THEOREM 4.1.11. *Let* $\mathcal{S}$ *be a semi-resolving set for* $\mathrm{PG}(2,q)$*,* $q \geq 4$*. Then* $|\mathcal{S}| \geq \min\{9q/4 - 3, \tau_2(\mathrm{PG}(2,q)) - 2\}$*.*

COROLLARY 4.1.12.

(i) If $q \geq 121$ *is a square prime power, then* $\mu_S(\mathrm{PG}(2,q)) = 2q + 2\sqrt{q}$*. Moreover, if* $q > 256$*, then semi-resolving sets attaining equality are the union of two punctured, disjoint Baer subplanes (cf. Proposition 4.1.4 (iii)).*

(ii) If $q = r^h$*,* $h \geq 3$ *odd, and* $r \geq 11$ *is an odd prime power (possibly a prime), then* $\tau_2(\mathrm{PG}(2,q)) - 2 \leq \mu_S(\mathrm{PG}(2,q)) \leq \tau_2(\mathrm{PG}(2,q)) - 1$*.*

A blocking semioval is a blocking set that has precisely one tangent line to it at each of its points. As a corollary of Theorem 4.1.10, we obtain a new lower bound on the size of a blocking semioval.

COROLLARY 4.2.3. *Let* $\mathcal{S}$ *be a blocking semioval in* $\mathrm{PG}(2,q)$*,* $q \geq 4$*. Then* $|\mathcal{S}| \geq 9q/4 - 3$*.*

The upper chromatic number of a hypergraph $\mathcal{H}$ is the maximum number $\mathrm{UCN}(\mathcal{H})$ of colors with which one can color the points of $\mathcal{H}$ without creating a rainbow hyperedge (a hyperedge is rainbow, if no two of its points have the same color). In Chapter 5 we study the upper chromatic number of projective planes considered as hypergraphs.

DEFINITION 5.1.4. *A coloring of a projective plane is* trivial*, if it contains a monochromatic double blocking set of size* τ_2*, and every other color class consists of one single point.*

A trivial coloring of a projective plane Π_q of order q gives that $\mathrm{UCN}(\Pi_q) \geq v - \tau_2(\Pi_q) + 1$, where $v = q^2 + q + 1$. Tuza and Bacsó [2] proved that if $\tau_2(\Pi_q) = 2(q+1) + c(\Pi_q)$, then $\mathrm{UCN}(\Pi_q) \leq q^2 - q - c(\Pi_q)/2 + o(\sqrt{q})$. We prove the following.

THEOREM 5.1.6. *Let Π_q be an arbitrary projective plane of order $q \geq 8$, and let $\tau_2(\Pi_q) = 2(q+1) + c(\Pi_q)$. Then*

$$\mathrm{UCN}(\Pi_q) < q^2 - q - \frac{2c(\Pi_q)}{3} + 4q^{2/3}.$$

THEOREM 5.1.7. *Let $v = q^2+q+1$. Suppose that $\tau_2(\mathrm{PG}(2,q)) \leq c_0 q - 8$, $c_0 < 8/3$, and let $q \geq \max\{(6c_0 - 11)/(8 - 3c_0), 15\}$. Then*

$$\mathrm{UCN}(\mathrm{PG}(2,q)) < v - \tau_2 + \frac{c_0}{3 - c_0}.$$

Note that $\frac{c_0}{3-c_0} < 8$.

THEOREM 5.1.8. *Let $v = q^2 + q + 1$, $q = p^h$, p prime, and suppose that $q > 256$ is a square, or $p \geq 29$ and $h \geq 3$ odd. Then*

$$\mathrm{UCN}(\mathrm{PG}(2,q)) = v - \tau_2 + 1,$$

and equality is reached only by trivial colorings.

Chapter 6 is devoted to the Zarankiewicz problem. $K_{n,m}$ denotes the complete bipartite graph in which the vertex classes have n and m elements, respectively, and C_k is the cycle of length k. Note that $K_{2,2}$ is isomorphic to C_4.

DEFINITION 6.1.1. *A bipartite graph $G = (A, B; E)$ is $K_{\alpha,\beta}$-*free *if it does not contain α nodes in A and β nodes in B that span a subgraph isomorphic to $K_{\alpha,\beta}$. We call $(|A|, |B|)$ the* size *of G. The maximum number of edges a $K_{\alpha,\beta}$-free bipartite graph of size (m, n) may have is denoted by $Z_{\alpha,\beta}(m,n)$, and is called a* Zarankiewicz number. *Graphs attaining equality are called* extremal.

Determining the exact value of $Z_{\alpha,\beta}(m,n)$ is very hard in general. The incidence graph of a finite projective plane is C_4-free and it is well-known to be extremal, so $Z_{2,2}(n^2 + n + 1, n^2 + n + 1) = (n^2 + n + 1)(n + 1)$ whenever a projective plane of order n exists. We mostly concentrate on the C_4-free case, and give two general upper bounds and some constructions arising from projective planes (more generally, block designs); furthermore, we extend Guy's table [8] of Zarankiewicz numbers for small parameters. Let us collect some of the consequences here.

THEOREM 6.2.5. *Let $n \geq 15$, and $0 \leq c \leq n/2$, $c \in \mathbb{Z}$. Then*

$$Z_{2,2}(n^2+n+1-c, n^2+n+1) \leq (n^2+n+1-c)(n+1).$$

Equality holds if and only if a projective plane of order n exists. Moreover, graphs attaining equality are subgraphs of a projective plane of order n.

THEOREM 6.2.8. *Let q be a square prime power, and let $c(c-1) < 2(\sqrt{q}+1)$, $0 \leq c \in \mathbb{Z}$. Then*

$$Z_{2,2}(q\sqrt{q}+1+c, q(q-\sqrt{q}+1)) = q(q\sqrt{q}+1) + c(q-\sqrt{q}+1).$$

COROLLARY 6.2.12. *Let $0 \leq c \in \mathbb{Z}$, n odd. Then*

$$Z_{2,2}\left(\frac{n(n-1)}{2}+c, n^2\right) \leq \frac{(n^2+2c-1)n}{2}.$$

Equality can be reached for all $0 \leq c \leq n+1$ if there exists a projective plane of order n with an oval (e.g., if n is a prime power).

COROLLARY 6.2.15. *Let $n \geq 2$ and $0 \leq c \in \mathbb{Z}$. Then*

$$Z_{2,2}(n^2+c, n^2+n) \leq n^2(n+1)+cn, \tag{6.3}$$

$$Z_{2,2}(n^2-n+c, n^2+n-1) \leq (n^2-n)(n+1)+cn, \tag{6.4}$$

$$Z_{2,2}(n^2-2n+1+c, n^2+n-2) \leq (n^2-2n+1)(n+1)+cn. \tag{6.5}$$

Equality can be reached in all three inequalities if a projective plane of order n exists and $c \leq n+1$, $c \leq 2n$, or $c \leq 3(n-1)$, respectively. Moreover, if $c \leq n+1$, or $c \leq 2n$ and $n > 4$, then graphs reaching the bound in (6.3) *or* (6.4)*, respectively, can be embedded into a projective plane of order n.*

Let us mention a result about a rather special case, which is interesting because it shows that if our C_4-free bipartite graph is of size $(v+1, v)$, where $v = n^2+n+1$ is the number of points in a projective plane of order n, then we cannot do anything better than adding a vertex of degree one to the incidence graph of a projective plane of order n. This is not true in general: the gap between $Z_{2,2}(m,m)$ and $Z_{2,2}(m+1,m)$ can be arbitrarily large as shown by equality in (6.3) of Corollary 6.2.15 (with $c = n$ and $n+1$).

COROLLARY 6.2.22. *Let $n \geq 2$. Then*

$$Z_{2,2}(n^2+n+2, n^2+n+1) \leq (n^2+n+1)(n+1)+1,$$

and equality holds if and only if a projective plane of order n exists.

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